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Non-perturbative effects in $AdS_5 \times S^5$ string theory and $d = 4$ SUSY Yang-Mills

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Abstract: We show that five-dimensional anti de-Sitter space remains a solution to low-energy type IIB supergravity when the leading higher-derivative corrections to the classical supergravity (which are non-perturbative in the string coupling) are included. Furthermore, at this order in the low energy expansion of the IIB theory the graviton two-point and three-point functions in $AdS_5 \times S^5$ are shown not to be renormalized and a precise expression is obtained for the four-graviton and related S-matrix elements. By invoking Maldacena's conjectured connection between IIB superstring theory and supersymmetric Yang–Mills theory corresponding statements are obtained concerning correlation functions of the energy-momentum tensor and related operators in the large-N Yang–Mills theory. This leads to interesting non-perturbative statements and insights into the rôle of instantons in the gauge theory.

Keywords: Nonperturbative Effects, String Duality, Conformal Field Models in String Theory, Superstrings and Heterotic Strings.

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1. Introduction

Recent results on D-brane black hole physics [1] have led to a very interesting conjecture by Maldacena [2] which proposes an exact correspondence between string theory on asymptotically Anti-de-Sitter (AdS) spaces, and certain quantum field theories living on the boundary of the AdS space. Although the direct evidence for this conjecture is sparse (it consists primarily of the identity of multiplicities of short representations of the AdS supergroup which are found in the two pictures) it leads to a beautifully consistent picture of the possible behavior of strongly coupled gauge theories[1]. This correspondence also explains the extrapolation of many D-brane black hole calculations beyond their apparent range of validity. One is therefore tempted to simply believe the conjecture and explore its implications for the properties of gauge theory and string theory.

In this paper we will use the Maldacena conjecture to explore nonperturbative properties of four-dimensional maximally supersymmetric ($\mathcal{N} = 4$) $SU(N)$ Yang-Mills theory in the large-N limit, This theory is related by the conjecture to type IIB superstring theory compactified on $AdS_5 \times S^5$. This maximally SUSY background is known to be a solution of classical type IIB supergravity and therefore it is a solution to low energy IIB superstring theory to lowest order in the inverse string tension, α'. We will exploit knowledge of certain nonperturbative terms in the IIB effective action [5, 6]

[1] Furthermore, an extension of the conjecture in [3] has led to the proof of a striking new result in large-N gauge theory [4].

to obtain nonperturbative information about the large-N gauge theory. Of particular interest will be the R^4 term [5] (where R is the ten-dimensional Riemann curvature) which arises at order α'^3 relative to the Einstein-Hilbert term. This term can be expressed as a particular contraction of four Weyl tensors and has a coefficient $f_4(\rho, \bar{\rho})$ which is an exactly known modular function of the complex scalar field $\rho = c^{(0)} + ie^{-\phi}$, where ϕ is the type IIB dilaton (so that the string coupling is $g_{st} = e^{\phi}$) and $c^{(0)}$ is the Ramond-Ramond ($R \otimes R$) scalar. Other interaction terms of the same dimension are also known. They are related to this term by supersymmetry.

According to Maldacena's conjecture, which we review below, the region of validity of the α' expansion translates in gauge theory into the region of large $g_{YM}^2 N$, where g_{YM} is the Yang-Mills coupling constant. One way of achieving this is to hold g_{YM} fixed and take $N \to \infty$, which is of relevance to the Matrix approach [7]. In the 't Hooft limit $g_{YM}^2 N = \hat{g}^2$ is fixed with $N \to \infty$ so that $g_{YM}^2 = \hat{g}^2/N \to 0$. The α' expansion is relevant in this limit only for strong coupling (large values of $\hat{g}$). The common domain of validity of the two expansions is that of low energy, perturbative, string theory. On the other hand, since we have nonperturbative information about string theory we can explore the gauge theory when N is large, even outside the range of validity of the 't Hooft expansion. The nonperturbative terms in string theory which we discuss, lead to contributions to gauge theory correlators which are of a particular order in the strong coupling expansion in powers of $(g_{YM}^2 N)^{-\frac{1}{2}}$, but nonperturbative in g_{YM} itself.

The first application of our results will be to prove that the $AdS_5 \times S^5$ solution is an exact solution of string theory when the $O(\alpha'^3)$ nonperturbative terms are included. This small step towards proving that it is a consistent background for the full string theory follows essentially from the fact that $AdS_5 \times S^5$ is conformally flat. Furthermore, our results are in accord with certain nonrenormalization theorems for two-point functions proven directly in the gauge theory [8] and suggest a new one for three-point functions.[2] These theorems are the direct analog of the familiar non-renormalization theorems for graviton two-point and three-point functions in ten-dimensional flat space superstring theory, here generalized (at $O(\alpha'^3)$) to the $AdS_5 \times S^5$ background. The R^4 term in IIB supergravity translates into an exact (in g_{YM}) formula for the connected four point function of stress tensors to next to leading order in the strong coupling expansion. We will see that the $SL(2, Z)$ duality symmetry of the IIB string theory translates into a precise statement concerning the way in which the corresponding modular transformations of $\mathcal{N} = 4$ Yang-Mills act on these correlation functions in the large-N limit. Related statements will be deduced for other correlation functions, notably the correlation of sixteen fermionic spin-half superpartners of the lagrangian density. Since the function f_4 (and its supersymmetric relatives) can be expanded for small g_{st} as an exact sum over D-instanton contributions in the string theory we are

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[2]Although the existence of three-point function nonrenormalization theorems is known to some experts, there is no systematic discussion of them in the literature. A recent paper [9] provides a proof of a subset of these theorems.

able to make precise statements about Yang-Mills instanton contributions to the gauge theory for certain classes of correlation functions in the small g_{YM} limit (with $g_{YM}^2 N$ fixed and large).

2. The type IIB — SYM dictionary

According to the interpretation of [2] given in [10, 11] there is a precise mapping of the S-matrix elements of ten-dimensional superstring theory in the $AdS_5 \times S^5$ background[3] to correlation functions of operators in the large-N Yang-Mills theory which lives on the boundary of AdS_5. One starts by solving the field equations deduced from the effective action for the string theory, $S[\Phi]$, with specified boundary conditions for the fields on the four-dimensional boundary of AdS_5, $\Phi|_{\partial(AdS_5)} = \tilde{\Phi}$. The expression $S[\tilde{\Phi}]$ is then interpreted as the generating functional for correlation functions of operators in the four-dimensional superconformal Yang-Mills theory living on the boundary. Thus, a correlation function of K operators has the form,

$$\frac{\delta}{\delta\tilde{\Phi}_1} \cdots \frac{\delta}{\delta\tilde{\Phi}_K} S[\tilde{\Phi}] \,, \tag{2.1}$$

where the operators are located at points $y_1, \ldots, y_K$ in the four-dimensional boundary. It is easy to see that this definition identifies the correlation functions with a quite precise analog of the K-particle S-matrix element in the superstring theory. That is, both the S-matrix of string theory in Minkowski space, and the objects studied in [10] and [11] are obtained by solving the effective equations of motion with boundary conditions at infinity. Note that in this analogy, "the incoming and outgoing on-shell states" are localized at asymptotic points labelled by y_k, rather than being momentum eigenfunctions. The on-shell condition determines the behaviour of the field as a function of the extra AdS_5 coordinate, U.

This identification emphasizes the holographic [12] nature of string theory and shows us that it is related to the well known fact that the only physical quantities in the theory are on shell S-matrix elements. In field theory, the S-matrix is computed in terms of the generating functional with sources which are nonvanishing only at infinity. The existence of a gauge invariant off shell continuation of Green's functions is intimately related to locality of the underlying theory: field theory is not holographic. Conversely, the holographic nature of string theory is implicit in the statement that only the on shell S-matrix is observable. This statement of the holographic principle in string theory is

[3] We will use the term S-matrix elements to refer to the effective action as a function of boundary values described by [10] and [11], even though wave packets in AdS space do not separate and the usual concept of an S-matrix is ambiguous. These are certainly limits of S-matrix elements in the asymptotically flat space which exist before the Maldacena limit is taken. We further believe that in perturbative string theory, these amplitudes will be expressed as expectation values of BRST invariant vertex operators in the $AdS_5 \times S^5$ σ model. We thank E.Witten and D.Freedman for discussions of this issue.

more general and more covariant than the Thorn-Susskind [13, 12] "wee parton" ansatz in the light cone gauge.

The correspondence between couplings in the Yang-Mills and string theories is given by

$$g_{YM}^2 = 4\pi g_{st}, \qquad \theta = 2\pi c^{(0)}\,, \tag{2.2}$$

so that

$$S \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} = \rho\,, \tag{2.3}$$

and the combination $g_{YM}^2 N$ is given in terms of the string theory parameters by

$$g_{YM}^2 N = L^4 \alpha'^{-2}\,, \tag{2.4}$$

where L is the radius of curvature of the AdS_5 space. The 32 components of the local supersymmetry charges in the IIB superstring translate into the 32 components of the rigid superconformal symmetry of the $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory. Although Maldacena's conjecture is supposed to be valid for any values of g_{YM}^2 and N, one can only do computations in certain limits. As is typical in dual situations, the regions of validity of gauge theory and string theory computations are complementary. Only quantities protected by nonrenormalization theorems can be easily computed in both languages. However, if we believe the conjecture, it immediately tells us many things about both theories. For example, we are led to believe that the gauge theory has a large N limit for any value of g_{YM}, not just in the 't Hooft regime. Weakly coupled string theory explores the large coupling limit of the planar gauge theory (large $g_{YM}^2 N$), while the α' expansion at arbitrary string coupling explores the large N limit at arbitrary Yang-Mills coupling, g_{YM}. Perturbative gauge theory computations (small $g_{YM}^2 N$) are valid only in the regime where string theory lives on a space-time of sub-stringy scale.

3. R^4 terms in IIB string theory on $AdS_5 \times S^5$

The part of the low-energy effective IIB supergravity action which will concern us has the form (in string frame)

$$S^{IIB} = \frac{1}{\alpha'^4} \int d^{10}x \sqrt{-G} (e^{-2\phi} R + k\alpha'^3 e^{-\phi/2} f_4(\rho, \bar\rho) R^4 + \cdots)\,, \tag{3.1}$$

where k is a known constant and we have indicated by $\cdots$ the terms which depend on any field other than the metric and the complex scalar ρ. The fact that there is no known action for the self-dual five-form field strength, F_5, will not concern us since we will only make use of its equations of motion, which are solved, in the AdS_5 background, by $F_5 = c\epsilon_{\mu_1 \ldots \mu_5}$, where the constant c determines the cosmological constant.

The R^4 term in (3.1) has a coefficient $f_4(\rho, \bar\rho)$ which is a non-holomorphic modular function of the scalar field as required by the $SL(2, Z)$ symmetry of the type IIB theory

[5]. It is given by the nonholomorphic Eisenstein series,

$$f_4(\rho,\bar\rho) = \sum_{(m,n)\neq(0,0)} \frac{\rho_2^{3/2}}{|m+\rho n|^3}\,, \tag{3.2}$$

which may be expanded for large $e^{-\phi}$ (small string coupling) as

$$\begin{aligned} e^{-\phi/2} f_4 \sim 2\zeta(3)e^{-2\phi} + \frac{2\pi^2}{3} + (4\pi)^{3/2} e^{-\phi/2}\,\cdot \\ \cdot \sum_{M>0} Z_M M^{1/2}\left(e^{-2\pi M(e^{-\phi}+ic^{(0)})} + e^{-2\pi M(e^{-\phi}-ic^{(0)})}\right)\left(1+o(e^{\phi}/M)\right). \end{aligned} \tag{3.3}$$

The first two terms of this expansion are the tree-level and one-loop contributions of string perturbation theory while the remaining terms are a sum over charge-M D-instanton contributions, each of which has an infinite power series of perturbative corrections (which are explicitly given in [5]). The coefficient Z_M is given by

$$Z_M = \sum_{m|M} \frac{1}{m^2}\,, \tag{3.4}$$

where $m|M$ denotes that the sum is over the divisors of M.

The symbol R^4 is used in (3.1) as a shorthand way of writing a particular contraction of four Riemann tensors which can be conveniently expressed as an integral over an auxiliary Grassmann variable θ which is a sixteen-component complex chiral $SO(9,1)$ spinor [14, 5],

$$R^4 = \int d^{16}\theta (R_{\theta^4})^4\,, \tag{3.5}$$

where

$$R_{\theta^4} \equiv \bar\theta\gamma^{\mu\nu\sigma}\theta\bar\theta\gamma^{\rho\tau}{}_{\sigma}\theta\, R_{\mu\nu\rho\tau}\,. \tag{3.6}$$

Here $\bar\theta = \gamma^0\theta$ and the ten-dimensional gamma matrices with world indices are defined by

$$\gamma^\mu = e_a{}^\mu \gamma^a\,, \tag{3.7}$$

where γ^a are the usual $SO(9,1)$ gamma matrices, $e_\mu{}^a$ $(m = 1,2,3,4)$ is the zehnbein and a is a ten-dimensional tangent-space index $(\mu, a = 0, 1, \cdots, 9)$. Equation (3.5) implies that R^4 transforms as a scalar density under general coordinate transformations.

We now wish to consider the compactification of the theory on $AdS_5 \times S^5$, which is known to be a solution to the classical low energy theory with a non-vanishing F_5. It is of great significance that the only components of the curvature which contribute in the expression R_{θ^4} are those of the Weyl tensor, $C_{\mu\nu\rho\tau}$. Since the $AdS_5 \times S^5$ is conformally flat it follows that the R^4 interaction vanishes in this background. At the risk of seeming pedantic, we will here demonstrate this explicitly. The curvature for this background (which is the product of two symmetric spaces) breaks up into two disjoint pieces associated with different directions. We shall use a tangent-space basis for the curvature (so that all the zehnbeins will drop out of (3.5)) in which the AdS_5 directions $a, b, c, d = 0, 1, 2, 3, 4$ are labelled M, N, P, Q so that,

$$R_{abcd} \equiv R^{(1)}_{MNPQ} = -L^{-6}(\eta_{MP}\eta_{NQ} - \eta_{MQ}\eta_{NP})\,, \tag{3.8}$$

where L is the AdS_5 length scale and η_{MP} is the signature $(4,1)$ Minkowski metric. The components in the S^5 directions $a, b, c, d = 5, 6, 7, 8, 9$ are labelled m, n, p, q and are given by,

$$R_{abcd} = R^{(2)}_{mnpq} = L^{-6}(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np})\,, \tag{3.9}$$

and all other components of the curvature vanish.

Substituting these expressions into (3.6) gives

$$\begin{aligned} L^6 R^{(0)}_{\theta^4} &= \bar\theta\gamma^{ABC}\theta\,\bar\theta\gamma_{ABC}\theta - \bar\theta\gamma^{abc}\theta\,\bar\theta\gamma_{abc}\theta \\ &\quad + \bar\theta\gamma^{AB}\gamma^c\theta\,\bar\theta\gamma_{AB}\gamma^c\theta - \bar\theta\gamma^A\gamma^{bc}\theta\,\bar\theta\gamma_A\gamma^{bc}\theta\,. \end{aligned} \tag{3.10}$$

This expression has a manifest symmetry under the $SO(4,1) \times SO(5)$ subgroup of $SO(9,1)$. In this decomposition the $SO(9,1)$ spinor is written as a bi-spinor in the $(4,4)$ representation so that $\theta \equiv \theta_{\alpha_1,\alpha_2}$ where the subscripts 1 and 2 label the $SO(4,1)$ spinor and the $SO(5)$ spinor, respectively. The symbol $\bar\theta$ is defined similarly for each of the two groups. For a $SO(4,1)$ spinor $\bar\phi_{\alpha_1} = (\phi\gamma^0_1)_{\alpha_1}$. For $SO(5)$ (which is pseudoreal) the bar is defined by $\bar\phi_{\alpha_2} = (\phi^* J)_{\alpha_2}$ with $JJ^\dagger = 1$. The similarity between the two groups is exhibited most clearly if a Majorana representation is chosen for $SO(4,1)$ in which all the gamma matrices, γ^M_1, are real and the same form is chosen for the gamma matrices, γ^m_2 $(m = 1, \cdots, 4)$, of $SO(5)$ but with $\gamma^0_2 = i\gamma^0_1$. In this basis the hermitian matrix J is given by $J = i\gamma^0_2$. We therefore conclude that in this basis,

$$\bar\theta = i\theta^*\gamma^0_1\gamma^0_2\,. \tag{3.11}$$

Substituting this into (3.10) it follows immediately that

$$R^{(0)}_{\theta^4} = 0\,, \tag{3.12}$$

since the difference in signature between $SO(4,1)$ and $SO(5)$ does not affect the contractions in (3.10). This, of course, is a particular property of the compactification on $AdS_5 \times S^5$ which follows from the fact that it is conformally flat (so the Weyl tensor vanishes) as mentioned earlier. This is not a property of more general backgrounds. [4]

This very simple result leads immediately to several important consequences. We see from (3.5) that $R^4 = 0$ in the $AdS_5 \times S^5$ background. This means that the dilaton equation of motion, which has a term proportional to R^4, is unchanged in this background. We also see that $\delta R^4/\delta g_{\mu\nu} = 0$ (since the differential is proportional to $(R^{(0)}_{\theta^4})^3$), which means that the Einstein equation is also unaffected. It is also easy to see that none of the other $O(\alpha'^3)$ terms, which are related to R^4 by supersymmetry,

[4] For example, compactification on Calabi-Yau threefolds has been considered in [15, 16].

contribute to the equations of motion. This means that the $AdS_5 \times S^5$ background is unaltered by the presence of this term, so this background is a solution of the effective equations of motion of string theory through $o(\alpha'^3)$, to all orders in g. As yet there is no σ-model argument which shows that the $AdS_5 \times S^5$ background is a solution of tree level string theory although this is undoubtedly true. Such arguments usually depend on world sheet superconformal invariance which is broken by the five form background.

Equation (3.12) also implies that

$$\frac{\delta}{\delta g_{\mu\nu}} \frac{\delta}{\delta g_{\rho\sigma}} R^4|_{AdS_5 \times S^5} = 0 = \frac{\delta}{\delta g_{\mu\nu}} \frac{\delta}{\delta g_{\rho\sigma}} \frac{\delta}{\delta g_{\tau\omega}} R^4|_{AdS_5 \times S^5} . \tag{3.13}$$

This shows that there is no renormalization of the graviton two-point or three-point functions in the $AdS_5 \times S^5$ background at $O(\alpha'^3)$. These are expected to extend to exact non-renormalization theorems analogous to those which are known to be true in string perturbation theory around ten-dimensional Minkowski space.

There is a non-zero four-graviton contribution from the R^4 term which is obtained by differentiating four times with respect to the metric, which leaves no overall powers of $R^{(0)}_{\theta 4}$. This adds to the classical term which arises from the Einstein-Hilbert action so that up to $O(\alpha'^3)$ the four-graviton amplitude is proportional to

$$e^{-2\phi} A_4^{String\,(1)} + k\alpha'^3 e^{-\phi/2} f(\rho, \bar\rho) A_4^{String\,(2)} , \tag{3.14}$$

where $A_4^{String\,(1)}$ is the classical amplitude obtained from the Einstein-Hilbert action while $A_4^{String\,(2)}$ is the contribution from the R^4 term. The new term can be computed in terms of the four-point vertex in the effective action, which is proportional to

$$S_4 = \alpha'^3 \int d^{10}x \sqrt{G} e^{-\phi/2} f(\rho, \bar\rho) t_8^{\mu_1\omega_1 \ldots \mu_4\omega_4} t_8^{\nu_1\tau_1 \ldots \nu_4\tau_4} R_{\mu_1\omega_1\nu_1\tau_1} \ldots R_{\mu_4\omega_4\nu_4\tau_4} . \tag{3.15}$$

The scalar field ρ is set to its constant background value while the linearized curvature is

$$R_{\mu_1\omega_1\nu_1\tau_1} = D_{\omega_1} D_{\tau_1} h_{\mu_1\nu_1} , \tag{3.16}$$

where $h_{\mu\nu}$ is the linearized fluctuation of the metric around its value in $AdS_5 \times S^5$ and D is the $AdS_5 \times S^5$ covariant derivative. The symmetries of $R_{\mu\omega\nu\tau}$ are imposed by the symmetries of the tensor t_8 (defined by eq. (2.16) of [17]) which has the form of the product of four inverse $AdS_5 \times S^5$ metrics summed over various permutations of their indices. The boundary Yang-Mills field theory that we are interested in will be obtained by substituting the solution of Einstein's equations linearized around $AdS_5 \times S^5$ for $h_{\mu_r\nu_r}$, with the boundary condition that it approaches the Minkowski plane wave with specified momenta, at infinity. In particular, correlations of the Yang-Mills stress tensor will arise from the components of $h_{\mu_r\nu_r}$ which are oriented in the four-dimensional Minkowski directions, $\mu_r = M_r$, $\nu_r = N_r$ where $M_r, N_r = 0, 1, 2, 3$. It is important that it is the ten-dimensional momenta which satisfy an on-shell constraint and not the four-dimensional Minkowski momenta, k_{M_r}. If we restrict consideration to s-waves with

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respect to the S^5 then only the AdS_5 part of the ten-dimensional metric is important and this has the form,

$$ds^2 = U^2(-dt^2 + (d\mathbf{x})^2) + \frac{dU^2}{U^2}\,, \tag{3.17}$$

where $\mathbf{x} = \{x_0, x_1, x_2, x_3\}$ and $U = x_4$. The on-shell condition determines the U dependence of $h_{\mu\nu}$ in terms of the Minkowski momenta. From these expressions we can, by Fourier transformation, obtain the objects which are supposed to match with local correlation functions of stress tensors in the supersymmetric Yang-Mills theory.

4. Non-perturbative terms in large-N SUSY Yang-Mills correlation functions

Using the Maldacena conjecture, our results for the string effective action in $AdS_5 \times S^5$ may be converted into statements about correlation functions in SYM theory. We will concentrate mainly on statements about the scattering of gravitons with polarizations in the "Minkowski"directions of AdS_5, which translate into properties of correlation functions of the SYM stress tensor, $T_{\mu\nu}$. Many other correlation functions are related to these by supersymmetry. The coefficient of the R^4 term in (3.1) has the prefactor $(e^{\phi/2}\alpha')^3 f(\rho) = L^6 N^{-3/2} f(S)$ relative to the Einstein-Hilbert term, which means that it is a non-leading term in the $1/N$ expansion.

The vanishing of the graviton one-point function, is simply the statement that the one-point function of the stress tensor $\langle T_{\mu\nu}\rangle$ vanishes, which follows from conformal invariance. Similarly, the vanishing of the R^4 contribution to the two-graviton S-matrix element in IIB supergravity translates into the statement that the correlation function of two stress tensors in $\mathcal{N} = 4$ Yang-Mills theory, $\langle T_{\mu_1\nu_1}(1) T_{\mu_2\nu_2}(2)\rangle$, is given by its free field value. This is known to be an exact statement [18, 8] by virtue of the relation of this correlation function to the R-symmetry anomaly, which is not renormalized due to the Adler-Bardeen theorem. An analogous Ward identity prevents the three-point correlation function of the stress tensor, $\langle T_{\mu_1\nu_1}(1) T_{\mu_2\nu_2}(2) T_{\mu_3\nu_3}(3)\rangle$ from receiving renormalizations beyond those of the free field theory[5] which is in accord with the fact that the three-graviton amplitude in string theory is not renormalized from its classical value. It is only when we come to the four-graviton amplitude that the R^4 term contributes and therefore the correlation function of four stress tensors gets a new contribution.

From the supergravity calculations in the last section we obtain the following expression for the momentum-space correlation function of four stress tensors in theYang-Mills theory,

$$A_4^{YM} = \langle T_{\mu_1\nu_1}(1) T_{\mu_2\nu_2}(2) T_{\mu_3\nu_3}(3) T_{\mu_4\nu_4}(4)\rangle = A_4^{(1)} + \tilde{k} N^{-3/2} f(S) A_4^{(2)} + \dots\,, \tag{4.1}$$

[5] We are grateful to Hugh Osborn for explaining this to us.

where $A_4^{(1)}$ is the contribution at leading order in $(g_{YM}^2 N)$ and $A_4^{(2)}$ is the correction arising from the R^4 term (and an irrelevant constant has been absorbed into $\tilde{k}$). In writing these equations we are using the correspondence between field theory correlators and string theory S-matrix elements. In particular, since we are using the effective action, we need only compute tree diagrams. The term $A_4^{(1)}$ comes from the four-graviton amplitude computed from the Einstein action in $AdS_5 \times S^5$, while $A_4^{(2)}$ corresponds to the four-graviton vertex in the R^4 term. No further terms are necessary because we are working to next to leading order in the α' expansion. The asymptotic states in both terms are taken to be delta functions in the boundary Minkowski space. The supergravity field ρ has here been interpreted as the complex coupling constant, $S = \theta/2\pi + 4\pi i/g_{YM}^2$. In writing (4.1) an overall factor of $\alpha'^{-4} = (g_{YM}^2 N)^2$ has been absorbed into the normalization of the stress tensors. In string theory scattering amplitudes, we are using the normalization in which all tree level amplitudes are of order one, while in the gauge theory we use the corresponding normalization in which all connected correlation functions of single trace operators are of order one in the large N limit. Terms of higher order in $(g_{YM}^2 N)^{-\frac{1}{2}}$ can be neglected if $1 >> \alpha'^{-2} L^4 = (g_{YM}^2 N)^{-1}$.

The second term in (4.1) has a remarkable amount of information concerning the Yang-Mills theory. It is the first non-leading term in the $1/N$ expansion but is an exactly known function of the (complex) coupling. The factor $f(\rho)$, defined in (3.2), has the expansion (3.3) for small g_{st} $(\rho \to \infty)$ which starts with the tree-level term $f \sim g_{st}^{-3/2} = (4\pi)^{3/2} g_{YM}^{-3}$. Therefore, in the limit $g_{YM} \to 0$ with $g_{YM}^2 N$ fixed and large, the expression (4.1) takes the form

$$A_4^{YM} = A_4^{(1)} + \tilde{k} A_4^{(2)} \Bigg[2\zeta(3) \left(\frac{g_{YM}^2 N}{4\pi}\right)^{-3/2} + \frac{2\pi^2}{3N^2} \left(\frac{g_{YM}^2 N}{4\pi}\right)^{1/2} + \frac{(4\pi)^{3/2}}{N^{3/2}} \sum_{M=1}^{\infty} Z_M M^{1/2} \left(e^{-M\left(8\pi^2 g_{YM}^{-2} + i\theta\right)} + e^{-M\left(8\pi^2 g_{YM}^{-2} - i\theta\right)} \right) \left(1 + o(g_{YM}^2/M)\right) \Bigg], \quad (4.2)$$

which includes an infinite series of instanton corrections. We will return to a discussion of these corrections in the next section.

It is important to note that S-duality (the Montonen-Olive duality), which is a discrete group which maps the small coupling regime to large coupling, is manifest in (4.1) but not in (4.2), which is the 't Hooft expansion, and is only valid when g_{YM} is very small. We remark in passing that our result does not agree with the general form suggested in a recent paper of Eguchi [19], who obtained constraints on the strong coupling behavior of the theory by insisting that duality be implemented in the 't Hooft expansion. When translated into string theory language, one of these constraints implies that the α' expansion at fixed g contains only even powers of α'. The Einstein and R^4 terms however differ by three powers of α'.[6]

[6] We thank Steve Shenker for a discussion of this point.

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5. Other non-perturbative contributions — Instanton effects

An easy way of determining all the interactions in the effective IIB supergravity action which are of the same dimension as R^4 and are related to it by supersymmetry is to use an on-shell linearized superspace formalism. Thus, the chiral superfield $\Phi(\theta)$ is a function of the same 16-component chiral Grassmann spinor which we introduced earlier. The chiral constraint is $\bar{D}\Phi = 0$ (where D is the supercovariant derivative) and the constraints $D^4\bar{\Phi} = \bar{D}^4\Phi = 0$ eliminate the auxiliary components of the field [20], which has an expansion in terms of the on-shell physical fields,

$$\Phi(\theta) = \rho + \theta\lambda + \bar{\theta}\gamma^{\mu\nu\rho}\theta G_{\mu\nu\rho} + \ldots + R_{\theta^4} + \ldots , \tag{5.1}$$

where λ is the complex spin-1/2 'dilatino', G is a complex combination of the $R \otimes R$ and $NS \otimes NS$ field strengths and the dots indicate a series of terms which terminates at the power θ^8. The (Weyl) curvature enters in R_{θ^4} which has four powers of θ. The general interaction of the type we are concerned with is given by an integral of the form $\int d^{16}\theta F[\Phi]$, which selects out terms with sixteen powers of θ. These are the terms which originate from the integration over the sixteen fermionic zero modes associated with the supersymmetries which are broken by the presence of a D-instanton. Among these is the R^4 term as well as many others. Notably, there is a sixteen-fermion term of the form $\int d^{10}x \det e \, e^{-\phi/2} f_{16}(\rho\bar{\rho})\lambda^{16}$ which was considered in detail in [6]. This is the analogue of the 't Hooft vertex in Yang-Mills theory. The symbol λ^{16} indicates the fully antisymmetrized product of the sixteen chiral spinor fields and the function f_{16} is given in [6] as

$$\begin{aligned} f_{16}(\rho, \bar{\rho}) &= \Gamma(27/2)\rho_2^{\frac{3}{2}} \sum_{(m,n)\neq(0,0)} \frac{(m+n\bar{\rho})^{24}}{|m+n\rho|^{27}} \\ &\sim \Gamma(27/2)\zeta(3)e^{-3\phi/2} + \Gamma(23/2)e^{\phi/2} \\ &\quad + \pi^{-\frac{1}{2}}(4\pi e^{-\phi})^{12} \sum_{M>0} Z_M M^{\frac{25}{2}} e^{-2\pi M(e^{-\phi}+ic^{(0)})}(1 + o(e^{\phi}/M)) , \end{aligned} \tag{5.2}$$

where the second line includes only the leading contributions for small $g_{st} = e^{\phi}$ to each term in the instanton sum (the anti-instantons are suppressed by powers of e^{ϕ}). We see that there are perturbative tree and one-loop terms in addition to the infinite set of D-instanton terms. The classical IIB supergravity is invariant under $SL(2,R)$ transformations, under which λ^{16} transforms by an arbitrary phase (assuming we are working with the scalars in the coset $SL(2,R)/U(1)$). However, the expression for f_{16} is a modular form with holomorphic and anti-holomorphic weights $(12, -12)$ which transforms with a discrete phase under $SL(2,Z)$,

$$f_{16} \to \left(\frac{c\rho + d}{c\bar{\rho} + d}\right)^{12} f_{16} , \tag{5.3}$$

(integer c and d), thereby cancelling only the subset of the transformations of λ^{16} which are in $SL(2,Z)$. This reflects the fact that in the string theory the continuous

$SL(2,R)$ is not a symmetry, even at tree level. Similar non-holomorphic modular forms are associated with other interactions that are related to R^4 by supersymmetry, such as $f_8 G^8$ and many others.

We now turn to the corresponding description of the $\mathcal{N}=4$ SUSY Yang-Mills theory. A charge M D-instanton contributes a weight $e^{-2\pi M/g_{st}}$ to an S-matrix element in the IIB theory. Using the dictionary, this translates into a contribution to the corresponding process in the Yang-Mills theory of a contribution with weight $e^{-8M\pi^2/g^2_{YM}}$, which is the contribution of a charge-M Yang-Mills instanton. This means that the instanton contributions to the R^4, λ^{16} and other IIB interactions discussed earlier must have a direct interpretation in large-N Yang-Mills theory. Intuitively, this is clear by considering the euclidean D3-brane/D-instanton configuration, which preserves half of the $\mathcal{N}=4$ supersymmetries. This system has a 'Higgs' branch in which the D-instanton is represented by a finite-sized Yang-Mills instanton in the D3-brane.

This argument for interpreting the D-instanton process as a Yang-Mills instanton effect in the boundary conformal Yang-Mills theory is further motivated by counting the fermionic zero modes. We have already seen that in the IIB theory there are sixteen fermionic zero modes in a D-instanton background which correspond to the broken supersymmetry transformations. This leads, to the R^4, λ^{16} and other terms. In the classical $\mathcal{N}=4$ Yang-Mills theory an instanton background has $8N$ fermion zero modes, $2N$ for each of the 4 adjoint Weyl fermions. However, our string computation is valid only in the region of large g^2N. Most of the classical zero modes are not related to symmetries. In particular, there are no discrete remnants of anomalous $U(1)$ symmetries which dictate the number of fermions in the 't Hooft interaction. The superpotential terms (in $\mathcal{N}=1$ language) of the $\mathcal{N}=4$ theory break all classical symmetries apart from the $SU(4)$ R symmetry. Thus, in planar perturbation theory around the instanton, the superpotential can convert the classical 't Hooft vertex into terms with different numbers of external fermion legs. The only zero modes which are protected are those which follow from supersymmetry.

Those superconformal generators which fail to annihilate the instanton lead to normalizable zero modes [21] . There are precisely sixteen such superconformal zero modes which means that there must be a correlation function of sixteen spin-1/2 operators in the Yang-Mills theory which have the $\tilde{\lambda}$'s as their fermionic sources (where $\tilde{\lambda}$ is the boundary value of the dilatino). These operators can be obtained by a supersymmetry transformation on $\mathrm{Tr}F^2$ and have the form $\Psi = \gamma^{\mu\nu}\mathrm{Tr}(F_{\mu\nu}\psi) + \psi^3$ terms (where we are using ten-dimensional notation to label the $\mathcal{N}=4$ fields). Our calculation determines the form of the Ψ^{16} vertex in the large g^2N limit.

As with the correlation of four stress tensors, this correlation function of sixteen fermionic operators, can be extracted from the IIB expression by using the dictionary and has the form,

$$N^{-3/2} f_{16}(S)\Psi^{16}\,. \tag{5.4}$$

The invariance of this expression under $SL(2,Z)$ transformations is again manifest.

The expansion (5.2) can now be used to expand this expression for small Yang-Mills coupling g_{YM}. Even though the individual Yang-Mills instanton terms are obviously highly suppressed since they have factors of $e^{-8\pi^2 M g_{YM}^{-2}}$ the full instanton sum is crucial for ensuring that the correlation functions transform with appropriate weight under $SL(2,Z)$ S-duality transformations.

The leading term in the small coupling limit in which $g_{YM}^2 N$ is fixed and large is again a term of order $(g_{YM}^2 N)^{-3/2}$, which comes from the tree-level term in (5.2) and is not suppressed by powers of N. Thus, the strong coupling expansion of the sum of planar diagrams will give a nonzero contribution to the Green's function of sixteen powers of $\gamma^{\mu\nu}\mathrm{Tr}(F_{\mu\nu}\psi)$. Note that unlike the 't Hooft interaction in gauge theories with less SUSY, this does not break any classical $U(1)$ symmetry. Indeed, the superpotential (in $\mathcal{N}=1$ language) of the $\mathcal{N}=4$ theory breaks all potentially anomalous symmetries at the classical level. The full classical global symmetry is the $SU(4)$ R symmetry and this is not anomalous. It is preserved by the Ψ^{16} term.

Since the Yang-Mills instantons do not break any symmetry of the gauge theory, and since the perturbation expansion in g_{YM} is only asymptotic, one might wonder how one could separate their contributions from ambiguities in the resummation of the perturbation series. The answer is again obtained by appealing to string theory. There it is known that all amplitudes are independent of the constant mode of the $R\otimes R$ scalar, to all orders in string perturbation theory. In the gauge theory, the constant mode of the scalar is just the θ parameter. String theory therefore predicts that SYM Green functions are independent of θ to all orders of the $1/N$ expansion, but should pick up e^{-N} contributions which are periodic in θ.

This prediction appears to be verifiable by direct calculation in Yang Mills theory. Indeed, it is well known that correlation functions are independent of θ in Yang-Mills perturbation theory. However in purely bosonic Yang Mills theory, it is expected [22] that the sum of all planar diagrams does depend on θ. What distinguishes the $\mathcal{N}=4$ theory is its conformal invariance. The Green functions should be completely determined in terms of anomalous dimensions and operator product coefficients. The perturbative series for these quantities are not expected to have renormalon singularities, and therefore, in each order of the 't Hooft expansion, these series should be convergent [23]. Consequently, finite orders in the $1/N$ expansion should be independent of θ for all g^2N. θ dependent corrections should be periodic and of order e^{-N}. This is precisely the form of the nonperturbative terms which are predicted by string theory via the Maldacena conjecture.

It would be very interesting to go further and make a more precise comparison between D-instanton and SYM instanton calculations. In particular, the exact form of the D-instanton contributions to the processes described earlier gives a precise prediction of the measure to be associated with an M-instanton contribution. However, we should remember that our string computation only determines the large $g_{YM}^2 N$ behavior of instanton amplitudes. Thus, we should only expect to match semiclassical Yang Mills calculations for quantities which obey some sort of nonrenormalization theorem.

6. Discussion

By combining nonperturbative information from the low energy IIB string-theory effective action with the Maldacena conjecture we have arrived at a number of perturbative and nonperturbative predictions about the maximally SUSY Yang Mills theory in four dimensions. For example, it leads to a simple explanation of the nonrenormalization theorem for two-point correlation functions of the stress tensor. It also gives a simple argument for a nonrenormalization theorem for three-point functions which is much more difficult to demonstrate directly and is only known to a few experts. It also suggests the θ independence of all Green's functions to all orders in the $1/N$ expansion of the Yang-Mills theory which agrees with the heuristic argument given in section 5. It is undoubtedly true, and would be interesting to demonstrate, that all of these results follow from supersymmetry considerations in the conformally invariant Yang-Mills theory which are the image of the powerful supersymmetry constraints in IIB supergravity.

In addition to these nonrenormalization conditions we have also obtained expressions for various four-point correlation functions in the theory. These expressions are nonperturbative in g_{YM} and exact through next to leading order in an expansion in $(g_{YM}^2 N)^{-\frac{1}{2}}$. While we have not displayed these expressions in detail in the Yang-Mills theory, they can easily be translated from the corresponding amplitudes in IIB supergravity in an $AdS_5 \times S^5$ background. This shows that superconformally invariant gauge theories contain exactly calculable but highly nontrivial correlation functions analogous to the Seiberg-Witten formulae of nonconformally invariant $\mathcal{N} = 2$ theories (which become trivial in the conformally invariant case). It would again be interesting to find derivations of these terms from superconformal Ward identities.

Of course, our nonperturbative string theoretic information is rather limited. It covers only certain terms which are 'protected' by supersymmetry in the sense that they are given by integration over half the superspace. Also, the mapping into the gauge theory is restricted to the strong coupling expansion (large $g_{YM}^2 N$) which corresponds to the α' expansion of string theory. In order to understand the weak coupling regime (small $g_{YM}^2 N$) it will be necessary to understand exact properties of the string theory S-matrix, which is somewhat more difficult.

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